Technische Universität Berlin
Institut für Mathematik

Berlin Mathematical School

Mathematical Physics I, Dr. Matteo Petrera, Dr. Raphael Boll WS 15/16

http://www3.math.tu-berlin.de/geometrie/Lehre/WS15/MP1/

Exercise Sheet 2 (28.10.15)

Due date: ***04.11.15***

- To get the Übungsschein (necessary condition for the oral exam) you need to collect *60% of the total sum of points*. Each exercise sheet has 30 points. The total number of sheets is 12-14.
- Please work in fixed groups of *2 students*.
- Please justify each step of your computations. *Results without any explanation are not accepted.* Please write in a readable way. *Unreadable handwriting will not be corrected.* Feel free to write your answers either in English or in German.
- Please turn in your homework directly to me at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed.*

Exercise 1 (3+1+1 pts)

Consider the following IVP in $\mathbb{R}$:

$$\begin{cases} \dot{x} = \dfrac{1}{2}\left(x^2 - 1\right), \\ x(0) = x_0 \in \mathbb{R}. \end{cases}$$

1. Find the solution.
2. Find the maximal intervals $(t_-, t_+) \subseteq \mathbb{R}$ on which the solution is defined.
3. What happens if the solution approaches t_- from the right and t_+ from the left?

Exercise 2 (2+2 pts)

Consider the following differential equation:

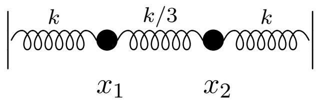
$$\dot{x} = -2tx^2 \tag{1}$$

1. Solve equation (1) (the solution will depend on one arbitrary constant $\alpha \in \mathbb{R}$).
2. Depending on α, find the maximal intervals for all solutions.

Turn over

Exercise 3 (2+4+1 pts)

Consider a one-dimensional system consisting of two masses m and three springs with elasticity constants k, $k/3$ and k, $k > 0$. The springs with elasticity constant k are firmly attached to the wall, while the spring with elasticity constant $k/3$ connects the two masses.



Denote by x_1 and x_2 the positions of the masses when the springs are in their rest points and consider the initial conditions

$$(x_1(0), x_2(0)) = (a, b), \qquad (\dot{x}_1(0), \dot{x}_2(0)) = (c, d),$$

with $b > a > 0$ and $c, d > 0$.

1. According to Hooke law, express the spring forces acting on each mass (frictional forces are neglected) and write down the coupled system of Newton equations of motion for the two masses.

   (Hint: Hooke law states that the spring force is $f(x) := -k\,x$, where $k > 0$ is the elasticity constant and x is the displacement of the end of the spring from its equilibrium position)

2. Decouple the obtained differential equations by introducing new variables $(\widetilde{x}_1, \widetilde{x}_2) := (x_1 + x_2, x_1 - x_2)$. Solve the IVP in the variables $\widetilde{x}_1$, $\widetilde{x}_2$.

3. Determine explicitly the flow of the system in terms of the variables x_1, x_2.

Exercise 4 (3+5 pts)

Consider the following IVP in $\mathbb{R}$:

$$\begin{cases} \dot{x} = x + f(t), \\ x(0) = x_0 \in \mathbb{R}, \end{cases}$$

where $f \in C^1(\mathbb{R}, \mathbb{R})$.

1. Find the solution.

2. Use the result in 1. to prove that the IVP of n ODEs

   $$\begin{cases} \dot{x}_k = x_{k-1} + x_k, \\ x_k(0) = 1, \end{cases}$$

   where $k = 1, \dots, n$ and $x_0 \equiv 0$, admits the solution

   $$x_k(t, 1) = \mathrm{e}^t \sum_{j=0}^{k-1} \frac{t^j}{j!}, \qquad k = 1, \dots, n.$$

   Compute the limit $\lim_{n\to\infty} x_n(t, 1)$.

Turn over

Exercise 5 (2 pts)

Consider a discrete dynamical system defined in terms of iterations of the map

$$\Phi : \mathbb{R}^2 \setminus \{(x_1, x_2) \in \mathbb{R}^2 \,:\, x_1 = 1/\alpha\} \to \mathbb{R}^2$$

defined by

$$(x_1, x_2) \mapsto \left(\frac{x_1}{\alpha\, x_1 - 1}, \frac{x_2 + \alpha\, x_1(x_1 - x_2)}{\alpha\, x_1 - 1} \right), \qquad \alpha > 0.$$

Explain why this discrete system exhibits a trivial behavior.

Exercise 6 (4 pts)

Let $(\Phi^t)_{t\in\mathbb{R}}$, with $\Phi^t : M \to M \subset \mathbb{R}^n$, be a continuous dynamical system. Consider a point $x_0 \in M$ and consider the orbit $\mathcal{O}(x_0) := \{\Phi^t(x_0) : t \in \mathbb{R}\} \subset M$. Prove that if $\mathcal{O}(x_0)$ is a closed orbit then all points $x \in \mathcal{O}(x_0)$ have the same period.